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On N Quasi D-Operator Operators

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Abstract: In this paper, we introduce the class of N quasi D-operator acting on the usual Hilbert space H over the complex plane. An operator T is said to be an N quasi D-operator if $T(T^{*2}(T^D)^2) = N(T^*T^D)^2T$, where N is a bounded operator on H. We investigate the basic behavior of this class of operator.

Keywords: Normal operators, D-Operator, Almost Class (Q), quasi-class (Q) operators, N quasi D-operator.

1. Introduction

H denotes the superable complex Hilbert space in this paper, while $B(H)$ is the usual Banach algebra of all bounded linear operators on H. Let $T \in B(H)$, Drazin inverse of T is an operator $T^D \in B(H)$, such that $TT^D = T^DT$, $T^D = T^DTT^D$ and $T^{k+1}T^D = T^k$ provided it exists. An operator $T \in B(H)$ is said to be D-Operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$ [1], class (Q) if $T^{*2}T^2 = (T^*T)^2$ [4], M Quasi class (Q) if $T(T^{*2}T^2) = M(T^*T)^2T$ [5], Quasi class (Q) if $T(T^{*2}T^2) = (T^*T)^2T$, N quasi D-Operator if $T(T^{*2}(T^D)^2) = N(T^*T^D)^2T$, for a bounded linear operator N. Let $T = \xi + i\zeta$, with $\xi = Re(T) = \frac{T^D+T^*}{2}$ and $\zeta = Im(T) = \frac{T^D-T^*}{2i}$. We shall simply denote $U^2 = (T^*T^D)^2$ and $V^2 = T^{*2}(T^D)^2$ where C and V are non-negative definite.

2. Main Results

Definition 2.1. *Let $T \in B(H)$ be Drazin invertible, an operator T is called N Quasi D-Operator if $T(T^{*2}(T^D)^2) = N(T^*T^D)^2T$ where N is a bounded operator on H.*

Theorem 2.2. *Let $T \in B(H)$ and let V commute with ξ and ζ such that $V^2T = NU^2T$, it follows that T is an N quasi D-operator.*

Proof. We recall that $T = \xi + i\zeta$, with $\xi = Re(T) = \frac{T^D+T^*}{2}$ and $\zeta = Im(T) = \frac{T^D-T^*}{2i}$ and $U^2 = (T^*T^D)^2$ and $V^2 = T^{*2}(T^D)^2$. Since $V\xi = \xi V$ and $U\zeta = \zeta U$, we have; $V^2\xi = \xi V^2$ and $U^2\zeta = \zeta U^2$, thus

$$V^2T + V^2(T)^* = TV^2 + (T)^*V^2$$
$$V^2T - V^2(T)^* = TV^2 - (T)^*V^2$$

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implies $TV^2 = V^2T$. Hence;

$$\begin{aligned} T(T^{*2}(T^D)^2) &= ((T^*(T^*T^D)T^D)T \\ &= (T^*T^D)^2T. \\ TU^2 &= NU^2T \\ \Rightarrow T(T^{*2}(T^D)^2) &= N((T^*(T^*T^D)T^D)T \\ T(T^{*2}(T^D)^2) &= N(T^*T^D)^2T \end{aligned}$$

Hence T is an N Quasi D-Operator. □

Proposition 2.3. *Let $T \in B(H)$ be a D-operator where $V^2\xi = \frac{1}{N}\xi V^2$ and $V^2\zeta = \frac{1}{N}\zeta V^2$, then T is an N Quasi D-Operator.*

Proof. $V^2\xi = \frac{1}{N}\xi V^2$ and $V^2\zeta = \frac{1}{N}\zeta V^2$ implies

$$\begin{aligned} V^2(\xi + i\zeta) &= \frac{1}{N}(\xi + i\zeta)V^2 \\ V^2T &= \frac{1}{N}TV^2 \\ (T^*(T^*T^D)T^D)T &= \frac{1}{N}T(T^*(T^*T^D)T^D) \\ T(T^*(T^*T^D)T^D) &= N(T^*(T^*T^D)T^D)T \\ &= N(T^*T^D)^2 \quad \text{(Since T is a D-operator).} \end{aligned}$$

Hence T is an N Quasi D-Operator. □

Theorem 2.4. *Let T_α and T_β be two N Quasi D-Operators from $B(H,H)$ such that $T_\alpha^D T_\beta^{*2} = T_\beta^D T_\alpha^{*2} = T_\alpha^{*2}(T_\beta^D)^2 = T_\beta^{*2}(T_\alpha^D)^2 = 0$, then $T_\alpha + T_\beta$ is an N Quasi D-Operator.*

Proof. Since T_α and T_β are N Quasi D-Operators, we have;

$$\begin{aligned} (T_\alpha + T_\beta)[(T_\alpha + T_\beta)^{*2}(T_\alpha^D + T_\beta^D)^2] &= (T_\alpha + T_\beta)[(T_\alpha^{*2} + T_\beta^{*2})((T_\alpha^D)^2 + (T_\beta^D)^2) \\ &= (T_\alpha + T_\beta)[T_\beta^{*2}(T_\alpha^D)^2 + T_\beta^{*2}(T_\beta^D)^2 + T_\alpha^{*2}(T_\alpha^D)^2 + T_\alpha^{*2}(T_\beta^D)^2] \\ &= (T_\alpha + T_\beta)[T_\beta^{*2}(T_\beta^D)^2 + T_\alpha^{*2}(T_\alpha^D)^2] \ \text{ since } T_\beta^{*2}(T_\alpha^D)^2 = T_\alpha^{*2}(T_\alpha^D)^2 = 0 \\ &= (T_\alpha + T_\beta)[T_\beta^{*2}(T_\beta^D)^2 + T_\alpha^{*2}(T_\alpha^D)^2] \\ &= T_\alpha T_\alpha^{*2}(T_\alpha^D)^2 + T_\beta T_\beta^{*2}(T_\beta^D)^2 \ \text{ since } T_\alpha T_\beta^{*2}(T_\beta^D)^2 = T_\beta T_\alpha^{*2}(T_\alpha^D)^2 = 0 \\ &= N(T_\alpha^{*2}(T_\alpha^D)^2)T_\alpha + N(T_\beta^{*2}(T_\beta^D)^2)T_\beta \\ &= N(T_\alpha^* T_\alpha^D)^2 T_\alpha + N(T_\beta^* T_\beta^D)^2 T_\beta \end{aligned}$$

Thus $T_\alpha + T_\beta$ is an N Quasi D-Operator. □

Theorem 2.5. *Let T_α and T_β be two N Quasi D-Operators from B(H,H) such that $T_\alpha^D T_\beta^{*2} = T_\beta^D T_\alpha^{*2} = T_\alpha^{*2}(T_\beta^D)^2 = T_\beta^{*2}(T_\alpha^D)^2 = 0$, then T_α - T_β is an N Quasi D-Operator.*

Proof. The proof follows from Theorem 2.4 above. □

Theorem 2.6. *Let T_α and T_β be two N Quasi D-Operators, then T_α T_β is an N Quasi D-Operator provided $T_\alpha T_\beta = T_\beta T_\alpha$ and $(T_\alpha^D)^2 T_\beta^{*2} = T_\beta^{*2}(T_\alpha^D)^2$.*

Proof. Since T_α and T_β are N Quasi D-Operators, we have;

$$\begin{aligned}
(T_\alpha T_\beta)[(T_\alpha T_\beta)^{*2}((T_\alpha T_\beta)^D)^2] &= (T_\alpha T_\beta)[(T_\alpha^{*2}T_\beta^{*2})(T_\alpha^D T_\beta^D)^2] \\
&= (T_\alpha T_\beta)[(T_\beta^{*2}T_\alpha^{*2})(T_\alpha^D T_\beta^D)^2] \\
&= T_\alpha(T_\beta T_\alpha^{*2})(T_\beta^{*2}(T_\alpha^D)^2)(T_\beta^D)^2 \\
&= T_\alpha(T_\alpha^{*2}T_\beta)(T_\beta^{*2}(T_\alpha^D)^2)(T_\beta^D)^2 \\
&= T_\alpha T_\alpha^{*2}T_\beta(T_\alpha^D)^2T_\beta^{*2}(T_\beta^D)^2 \\
&= T_\alpha T_\alpha^{*2}(T_\alpha^D)^2T_\beta T_\beta^{*2}(T_\beta^D)^2 \\
&= N(T_\alpha^{*2}(T_\alpha^D)^2)T_\alpha N(T_\beta^{*2}(T_\beta^D)^2)T_\beta \\
&= N(T_\alpha^{*2}((T_\alpha^D)^2T_\alpha)(T_\beta^{*2}(T_\beta^D)^2))T_\beta) \\
&= N(T_\alpha^{*2}(T_\alpha^D)^2T_\beta^{*2}T_\alpha(T_\beta^D)^2T_\beta) \\
&= N(T_\alpha^{*2}T_\beta^{*2}(T_\alpha^D)^2(T_\beta^D)^2T_\alpha T_\beta) \\
&= N[(T_\alpha T_\beta)^{*2}(T_\alpha^D T_\beta^D)^2(T_\alpha T_\beta)] \\
&= N[(T_\alpha T_\beta)^{*2}((T_\alpha T_\beta)^D))^2(T_\alpha T_\beta)] \\
&= N[(T_\alpha T_\beta)^*(T_\alpha T_\beta)^D]^2(T_\alpha T_\beta)
\end{aligned}$$

Thus $T_\alpha T_\beta$ is N Quasi D-Operator. □

Theorem 2.7. *Power of N Quasi D-operator is similarly N Quasi D-operator.*

Proof. We first show that the result holds for some $p = 1$, then we have ;

$$T(T^{*2}(T^D)^2) = N(T^*T^D)^2T \tag{1}$$

Suppose the result holds for $p = n$, we have;

$$[T(T^{*2}(T^D)^2)]^n = (N(T^*T^D)^2T)^n \tag{2}$$

We then prove that the result is true for $p = n + 1$. We have;

$$[T(T^{*2}(T^D)^2)]^{n+1} = (N(T^*T^D)^2T)^{n+1} \tag{3}$$

$$[T(T^{*2}(T^D)^2)]^{n+1} = [NT(T^{*2}(T^D)^2)]^n[NT(T^{*2}(T^D)^2)] \tag{4}$$

$$= [N(T^*(T^D))^2T]^n[N(T^*(T^D))^2T] \text{ by (1) and (2)}$$

$$[T(T^{*2}(T^D)^2)]^{n+1} = [N(T^*(T^D))^2T]^{n+1} \tag{5}$$

Hence the proof as required. □

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