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On Class (Q^*)

Wanjala Victor[1,*] and A. M. Nyongesa[1]

1 Department of Mathematics, Kibabii university, Bungoma, Kenya.

Abstract: Class (Q) operators was introduced and studied by Jibril in (quote). In this paper , we introduce and study the "cousins" to this class, namely class (Q^*). An operator $T \in B(H)$ is said to belong to class (Q^*) if $T^{*2}T^2 = (TT^*)^2$. We study the algebraic properties of this class. We also strike the relationship between this class and square hyponormal operators through characterization of (α, β)-class (Q) operators.

Keywords: Normal operators, Square-hyponormal operators, Class (Q) operators, Almost Class (Q), Class (Q^*) operators.

1. Introduction

H denotes the separable complex Hilbert space in this paper, while the Banach algebra of all bounded linear operators on H are denoted by $B(H)$. An operator $T \in B(H)$ is said to be normal if $T^*T = TT^*$, 2-normal if $T^*T^2 = T^2T^*$, quasinormal if $TT^*T = T^*T^2$, square normal if $T^{*2}T^2 = T^2T^{*2}$, square hyponormal if $(T^*T)^2 \geq (TT^*)^2$ [?], Class (Q) if $T^{*2}T^2 = (T^*T)^2$ [?], (α, β)-class (Q) if $\alpha^2 T^{*2}T^2 \leq (T^*T)^2 \leq \beta^2 T^{*2}T^2$ for $0 \leq \alpha \leq 1 \leq \beta$. (α, β)-class (Q) operators was covered by Wanjala Victor and A. M. Nyongesa [?]. We state the following well known results:

We first start by showing that this class is different from class (Q), this is illustrated in the following example. We then proceed to look at some nice properties of this class.

Example 1.1. *We consider* $T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $T^* = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. *A simple computation shows* $T^{*2}T^2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, $(T^*T)^2 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ *and* $(TT^*)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. *It follows that* $T^{*2}T^2 = (TT^*)^2$ *hence* $T \in (Q^*)$, *but* $T^{*2}T^2 \neq (TT^*)^2$ *hence* T *is not in class (Q).*

Corollary 1.2. *Let T be an operator on a Hilbert space H. Then T^* is also an operator on H and the following properties hold:*

(1). $\| T^* \| = \| T \|$.

(2). $(T^*)^* = T$.

Corollary 1.3. *Let T be an operator. Then*

(1). $\| T^*T \| = \| TT^* \| = \| T \|^2$.

(2). $T^*T = 0$ *if and only if* $T = 0$.

* E-mail: wanjalavictor421@gmail.com

2. Main Results

Definition 2.1. *Let $T \in B(H)$, then $T \in (Q^*)$ if $T^2 T^{*2} = (T^*T)^2$.*

Proposition 2.2. *Let T be an operator, then $\| T^*T \| = \| TT^* \| = \| T^* \|^2$.*

Proof. With $\| T^* \| = \| T \|$ from (i) of Corollary 1.2, we obtain; $\| T^*T \| \leq \| T^* \| \| T^* \| = \| T^* \|^2$ hence; $\| T^*T \| \leq \| T^* \|^2$. Similarly

$$\begin{aligned} \| T^*\xi \|^2 &= (T^*\xi, T^*\xi) \\ &= (TT^*\xi, \xi) \\ &\leq \| TT^*\xi \| \| \xi \| \\ &\leq \| TT^* \| \| \xi \|^2 \end{aligned}$$

Hence $\| T^* \|^2 \leq \| TT^* \|$. So $\| T^* \|^2 = \| TT^* \|$, replacing T by T^*, we get; $\| T^* \|^2 = \| T \|^2 = \| T^*T \|$ as required. □

Corollary 2.3. *Let $T \in B(H)$, if $T \in (Q^*)$ such that T^2 is normal, then T is normal.*

Proof. If T^2 is normal then by Proposition 2.2, T is a 2-normal operator. □

Proposition 2.4. *Let $T \in B(H)$, if $T \in (Q^*)$, then the following hold:*

(1). T^{*2} is in (Q^*).

(2). $T^{-1} \in (Q^*)$ provided it exists.

(3). Any $S \in B(H)$ that is unitarily equivalent to T is also in (Q^*).

Proof.

(1). Since $T \in (Q^*)$, $T^2 \in (Q^*)$, hence $(T^2)^* = (T^*)^2 = T^{*2} \in (Q^*)$.

(2). $T \in (Q^*)$ implies $T^2 \in (Q^*)$. Thus $(T^2)^{-1} = (T^{-1})^2 \in (Q^*)$.

(3). Suppose $T \in (Q^*)$ and S is unitarily equivalent to T. Then there exists a unitary operator $U \in B(H)$ such that $S^2 = UT^2U^*$ and $S^{*2} = UT^{*2}U^*$. Since $T^{*2} \in (Q^*)$, then $S^{*2} \in (Q^*)$ hence $S \in (Q^*)$. □

Proposition 2.5. *Let $T \in (Q^*)$, then If $(TT^*)^2 = T^{*2}T^2$.*

Proof. Since $T \in (Q^*)$, by Proposition 1.2, $T^* \in (Q^*)$. Hence, we have

$$\begin{aligned} ((T^*)^*(T^*))^2 &= (T^*)^2(T^*)^{*2} \\ &= (TT^*)^2 \\ &= T^{*2}T^2. \end{aligned}$$

□

Proposition 2.6. *Let $T, T^* \in B(H)$, if both of them are quasinormal, then $T \in (Q^*)$.*

Proof. T being quasinormal implies;

$$TT^*T = T^*T^2 \tag{1}$$

Similarly T^* being quasinormal implies;

$$T^*TT^* = TT^{*2} \tag{2}$$

From (1) and (2), we have

$$TT^*T = T^2T^* \tag{3}$$

Post multiplying (3) by T^* and post-multiplying by the same, we have;

$$\begin{aligned} TT^*TT^* &= T^2T^*T^* \\ &= T^{*2}T^2 \\ &= T^2T^{*2} \\ &= (TT^*)^2. \end{aligned}$$

□

Proposition 2.7. *If $T \in B(H)$ is both 2-Normal and in (Q^*), then $T \in (Q)$.*

Proof.

$$\begin{aligned} (TT^* - T^*T)^*(TT^* - T^*T) &= (TT^* - T^*T)(TT^* - T^*T) \\ &= (TT^*)^2 - TT^{*2}T - T^*T^2T^* + (T^*T)^2 \\ &= (TT^*)^2 - T^{*2}T^2 - T^2T^{*2} + (T^*T)^2 \quad (\text{Since } T \in 2N) \\ &= (TT^*)^2 - T^{*2}T^2 - T^2T^{*2} + (T^*T)^2 \quad (\text{since } T \in (Q^*)) \\ &= T^2T^{*2} - (T^*T)^2 \\ T^2T^{*2} &= (T^*T)^2 \end{aligned}$$

Hence $T \in (Q)$. □

Corollary 2.8. *If $T \in (Q^*)$, then $(T^*T)^2 = T^2T^{*2}$.*

Proof. $T \in (Q^*)$ implies $T^* \in (Q^*)$. Hence; $(T^*(T^*)^*)^2 = (T^*)^{*2}(T^*)^2 = (T^*T)^2 = T^2T^{*2}$. □

Definition 2.9. *An operator is said to be in (α, β)-Class (Q^*) if $\alpha^2T^{*2}T^2 \leq (TT^*)^2 \leq \beta^2T^{*2}T^2$ for $0 \leq \alpha \leq 1 \leq \beta$.*

Theorem 2.10. *Let $T \in B(H)$ be such that its in both (α, β)-Class (Q) and (α, β)-Class (Q^*), then T is a square-hyponormal operator.*

Proof. Suppose T is in both (α, β)-Class (Q) and (α, β)-Class (Q^*), then;

$$\alpha^2T^{*2}T^2 \leq (T^*T)^2 \leq \beta^2T^{*2}T^2. \tag{4}$$

Similarly, let $T \in (\alpha, \beta)$-Class (Q^*), then;

$$\alpha^2T^{*2}T^2 \leq (TT^*)^2 \leq \beta^2T^{*2}T^2. \tag{5}$$

Setting α to be 1 in (4)

$$(T^*T)^2 \geq T^{*2}T^2 \tag{6}$$

and setting β to be 1 in (5)

$$(TT^*)^2 \leq T^{*2}T^2 \tag{7}$$

From (6) and (7), we have

$$(T^*T)^2 \geq T^{*2}T^2 \geq (TT^*)^2,$$

which implies; $(T^*T)^2 \geq (TT^*)^2$ as required. ☐

Lemma 2.11. *If $T \in (Q^*)$, then its a square-normal operator.*

Proof. The proof follows directly from Corollary 1.3. ☐

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