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Further Generalization of Unitary Quasi-Equivalence of Operators

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Abstract: In this paper, we further generalize the class of n-Unitary Quasi-Equivalence by extending this study to (n,m)-Unitary Quasi-Equivalence. We investigate the properties of this class and also the relation of this equivalence class to other relations.

Keywords: Unitary Quasi-Equivalence , (n,m)-metric equivalence, (n,m)-unitary Quasi-Equivalence, (n,m)-normal operators.

1. Introduction

H is a superable complex Hilbert space and $B(H)$ is the Banach algebra of all bounded linear operators throughout this paper. $T \in B(H)$ is normal if $T^*T = TT^*$, n-normal if $T^*T^n = T^nT^*$, (n,m)-normal if $T^{*m}T^n = T^nT^{*m}$ projection if $T^2 = T$, Hyponormal if $T^*T \geq TT^*$, quasinormal if $T(T^*T) = (T^*T)T$, n-hyponormal if $T^*T^n \geq T^nT^*$, (n,m)-hyponormal if $T^{*m}T^n \geq T^nT^{*m}$. $S, T \in B(H)$ are said to be Metrically equivalent if $S^*S = T^*T$ [3], n-metrically equivalent if $S^*S^n = T^*T^n$ [6] and (n,m)-metrically equivalent if $S^{*m}S^n = T^{*m}T^n$ for more see [7], Unitarily Quasi-Equivalent if there exists a unitary operator $U \in B(H)$ such that $S^*S = UT^*TU^*$ and $SS^* = UTT^*U^*$ [1], n-unitarily quasi-equivalent if $S^*S^n = UT^*T^nU^*$ and $S^nS^* = UT^nT^*U^*$. Two operators S $\in B(H)$ and $T \in B(H)$ are said to be (n,m)-Unitarily Quasi-Equivalent if there is existence of a unitary operator $U \in B(H)$ such that $S^{*m}S^n = UT^{*m}T^nU^*$ and $S^nS^{*m} = UT^nT^{*m}U^*$ for positive integers n and m. We note that (n,m)-Unitarily Quasi-Equivalent operators are n-Unitarily Quasi-Wquivalent operators when $m = 1$ and Unitarily quasi equivalent when $n = m = 1$.

2. Main Results

Theorem 2.1. *(n,m)-Unitary Quasi-Equivalence is an equivalence relation.*

Proof. Suppose $S, T, P \in B(H)$, then S is (n,m)-Unitarily Quasi-Equivalent to S since $S^{*m}S^n = IS^{*m}S^nI^*$ and $S^nS^{*m} = IS^nS^{*m}I^*$ for $I = U$. If S is (n,m)-Unitarily Quasi-Equivlent to T, then $S^{*m}S^n = UT^{*m}T^nU^*$ and $S^nS^{*m} = UT^nT^{*m}U^*$. Pre-multiplying and post-multiplying the two equations by U^* and U on both sides we end up with $T^{*m}T^n = US^{*m}S^nU^*$ and $T^nT^* = US^nS^{*m}U^*$. Hence T is n-Unitarily Quasi-Equivalent to S. We now have to show that if S is n-Unitarily Quasi-Equivalent to T and T is n-Unitarily Quasi-Equivalent to P, then S is n-Unitarily Quasi-Equivalent to P. Now

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$S^{*m}S^n = UT^{*m}T^nU^*$ and $S^nS^{*m} = UT^nT^{*m}U^*$ and $T^{*m}T^n = VP^{*m}P^nV^*$ and $T^nT^{*m} = VP^nP^{*m}V^*$ where U and V are unitary operators. Then $S^{*m}S^n = UT^*T^nU^* = UVP^{*m}P^nV^*U^* = QP^{*m}P^nQ^*$, for $Q = UV$ which is unitary. Equally $S^nS^{*m} = UT^nT^{*m}U^* = UVP^nP^{*m}V^*U^* = QP^nP^{*m}Q^*$, for $Q = UV$ which is unitary. This shows that S is n-Unitarily Quasi-Equivalent to P and hence n-Unitary Quasi-Equivalence is an equivalence relation. □

Theorem 2.2. *Let $S, T \in B(H)$ be (n ,m)-unitarily Quasi-equivalent. Then S is (n,m)-normal if and only if T is (n,m)-normal.*

Proof. Suppose that S and T are (n,m)-unitarily quasi-equivalent and also suppose that S is (n,m)-normal, then $T^{*m}T^n = US^{*m}S^nU^*$ and $T^nT^{*m} = US^nS^{*m}U^*$. Hence $T^{*m}T^n = US^{*m}S^nU^* = US^nS^{*m}U^* = T^nT^{*m}$. The converse is proved in the similar way. □

Lemma 2.3. *Two operators $S, T \in B(H)$ are (n,m)-unitarily Quasi-equivalent if and only if $S^{*m}S^n - S^nS^{*m} = U(T^{*m}T^n - T^nT^{*m})U^*$.*

Theorem 2.4. *Let $S, T \in B(H)$ be (n,m)-unitarily Quasi-equivalent. Then S is (n,m)-hyponormal if and only if T is (n,m)-hyponormal.*

Proof. The proof follows from Lemma , $S^{*m}S^n - S^nS^{*m}$ is unitarily equivalent to $T^{*m}T^n - T^nT^{*m}$. If $S^{*m}S^n - S^nS^{*m} \geq 0$ then $T^{*m}T^n - T^nT^{*m} = U(S^{*m}S^n - S^nS^{*m})U^* \geq 0$. This shows that (n,m)-Unitary quasi-equivalence preserves (n,m)-hyponormality. □

We note that (n,m)-unitary quasi-equivalence preserves (n,m)-quasinormality and (n,m)-binormality of operators, this follows from Theorem 2.4 and the fact that these classes are contained in the class of (n,m)-hyponormal operators.

Lemma 2.5. *$T \in B(H)$ is (n,m)-unitarily equivalent to a unitary operator if and only if it is a unitary operator.*

Proof. Suppose that $T^n = PU^nP^*$, where $U, P \in B(H)$ are unitary operators. Then we have;

$$T^{*m}T^n = PU^{*m}P^*PU^nP^* = I$$

and

$$T^nT^{*m} = PU^nP^*PU^{*m}P^* = I$$

□

Lemma 2.5 can be extended to the class of (n,m)-unitarily Quasi-equivalent of operators.

Theorem 2.6. *$T \in B(H)$ is n-unitarily Quasi-equivalent to a unitary operator if and only if it is a unitary operator.*

Proof. Let $T \in B(H)$ is (n,m)-unitarily Quasi-equivalent to a unitary operator $P \in B(H)$, then there exists a unitary operator $U \in B(H)$ such that $T^{*m}T^n = U(P^{*m}P^n)U^* = I$ and $T^nT^{*m} = U(P^nP^{*m})U^* = I$. This implies that $T^{*m}T^n = T^nT^{*m}$. The converse follows from Lemma 2.5. □

Theorem 2.7. *If $S, T \in B(H)$ are both 2-Self adjoint and (2,2)-Unitarily quasi-equivalent, then S^4 and T^4 are unitarily equivalent.*

Proof. The proof follows directly from the definitions; $S^{*2}S^2 = UT^{2*}T^2U^*$ and $S^2S^{*2} = UT^2T^{*2}U^*$, then using the self adjoint property of S and T we have; $S^4 = UT^4U^*$. Hence the proof. □

Theorem 2.8. *Let $S, T \in B(H)$ be (n,m)-unitarily Quasi-equivalent. Then $\|S^n\| = \|T^n\|$.*

Proof. $\|S^n\|^2 = \|S^{*m}S^n\| = \|UT^{*m}T^nU^*\| = \|T^{*m}T^n\| = \|T^n\|^2$. Taking square root on both sides of the equation we get the intended result. □

Proposition 2.9. *Let $T \in B(H)$, then we have*

(1). $Ker(T^{*m}T^n) = Ker(T^n)$.

(2). $\overline{Ran(T^nT^{*m})} = \overline{Ran(T^n)}$.

Proof.

$$\begin{aligned}(1).\ Ker(T^{*m}T^n) &= \{\xi \in H : T^{*m}T^n\xi = 0\}\\ &= \{\xi \in H : T^n\xi = 0\}\\ &= Ker(T^n)\end{aligned}$$

$$\begin{aligned}(2).\ \overline{Ran(T^nT^{*m})} &= \overline{\{\xi \in H : \xi = T^nT^{*m}x, x \in H\}}\\ &= \overline{\{\xi \in H : \xi = T^n(T^{*m}x)\}}\\ &= \overline{Ran(T^n)}.\end{aligned}$$

□

Theorem 2.10. *If $S, T \in B(H)$ are (n,m)-unitarily Quasi-equivalent, then $Ker(S^n) = Ker(T^n)$ and $\overline{Ran(\|S^n\|)} = \overline{Ran(\|T^n\|)}$.*

Proof. The proof follows from Proposition 2 and the definition of n-unitary quasi-equivalence of operators. □

Corollary 2.11. *If $S, T \in B(H)$ are (n,m)-unitarily Quasi-equivalent and S^n is injective, then T^n is injective.*

We note that n-unitarily quasi-equivalence unlike n-metric equivalence preserves injectivity of operators.

References

[1] Benard Mutuku Nzimbi and Stephen Wanyonyi Luketero, *On unitary Quasi-Equivalence of Operators*, International Journal of Mathematics And its Applications, 8(1)(2020), 207-215.

[2] A. A. Jibril, *On n-power normal operators*, The Arabian Journal for Science and Engineering, 33(2008), 247-251.

[3] B. M. Nzimbi, G. P. Pokhariyal and S. K. Moindi, *A Note on Metric Equivalence of some Operators*, Far East J. F Math. Sci., 75(2013), 301-318.

[4] Mahmoud Kutkut, *On quasiequivalent operators*, Bull. Cal. Math. Soc., 90(1998), 45-52.

[5] S. A. Alzuraiqi and A. B. Patel, *On n-Normal Operators*, General mathematics Notes, 1(2)(2010), 61-73.

[6] Wanjala Victor, R. K. Obogi and M. O. Okoya, *On N-Metric equivalence of Operators*, International Journal of Mathematics And its Applications, 8(1)(2020), 107-109.

[7] Wanjala Victor and A. M. Nyongesa, *On (n,m)-metrically equivalent operators*, International Journal of Mathematics And its Applications, 9(2)(2021), 101-104.